**FINAL: MMATH LINEAR ALGEBRA**

Date: **20th November 2019**
The Total points is **110** and the maximum you can score is **100** points.

All vector spaces considered below are assumed to be finite dimensional.

(1) (7+7+7+7=28 points) Answer the following multiple choice questions about each of them. Write all correct options. No justification needed. **No partial credit will be given if a correct option is missing or an incorrect option is written.**

(a) Which of the following statements are true?
 (i) The set of complex normal matrices form a group under multiplication.
 (ii) The set of $n \times n$ real orthogonal matrices form a subgroup of the group of $n \times n$ complex unitary matrices.
 (iii) The set of $n \times n$ complex orthogonal matrices form a subgroup of the group of $n \times n$ complex unitary matrices.
 (iv) The group of $1 \times 1$ unitary matrices $U(1)$ is isomorphic to the group of $2 \times 2$ simple real orthogonal matrices $SO_2(\mathbb{R})$.

(b) Which of the following square matrices are diagonalizable?
 (i) A matrix whose characteristic polynomial is same as the minimal polynomial.
 (ii) A complex matrix $A$ such that $A^*A = AA^*$.
 (iii) A complex matrix whose minimal polynomial has distinct roots.
 (iv) A $n \times n$ matrix which has $n$ linearly independent eigen vectors.

(c) Let $R$ be a ring and $0 \to A \to B \to C \to 0$ be a short exact sequence of $R$-modules. Let $M$ be a $R$-module. The sequence

$$0 \to Hom_R(M, A) \to Hom_R(M, B) \to Hom_R(M, C) \to 0$$

is exact if
 (i) $M$ is a projective $R$-module.
 (ii) $A$ is a projective $R$-module.
 (iii) $B$ is a projective $R$-module.
 (iv) $C$ is a projective $R$-module.

(d) Let $R$ be a principal ideal domain. Which of the following are true.
 (i) Every submodule of a free $R$-module is torsion free.
 (ii) Every torsion free submodule of a finitely generated $R$-module is free.
 (iii) Every torsion free $R$-module is free.
 (iv) Every ideal of $R$ is a free $R$-module.

(2) (8+8+8+8+8=40 points) Prove or disprove (using a counterexample) the following statements.
   (a) Let $\phi$ and $\psi$ be two commuting diagonalizable linear operators of a vector space $V$. Then $\phi$ and $\psi$ are simultaneously diagonalizable.
   (b) Let $A$ and $B$ be $n \times n$ matrices such that minimal polynomial of $A$ and $B$ are equal and has degree $n$. Then $A$ and $B$ are similar matrices.
   (c) Let $R$ be an integral domain and $M$ be any $R$-module. The $R$-module $Hom_R(M, R)$ is torsion free.
   (d) Let $R$ be an integral domain and $M$ be a finitely generated torsion free $R$-module then $M$ is a free $R$-module.
   (e) Let $R$ be a commutative and $M$ be an $R$-module. Let $\phi : M \to M$ be an $R$-linear map. Then $\phi$ extends to a $R$-algebra homomorphism from the tensor algebra $T(M)$ to the symmetric algebra $S(M)$.

(3) (5+10=15 points) Let $(V, <, >)$ be a hermitian space. When is a linear operator on $V$ called non-negative operator? Let $A$ be a $n \times n$ hermitian matrix. Then show that $\exp(A)$ is a non-negative operator on the hermitian space $\mathbb{C}^n$ with the standard hermitian form.

(4) (15 points) Compute and list down rational canonical form and Jordan form of all the $4 \times 4$ real matrices whose minimal polynomial is $x^2 - 4$.

(5) (12 points) Let $R$ be an integral domain and $M$ be an $R$-module. Let $m_1, \ldots, m_n$ be generators of $M$ as an $R$-module. Suppose there exist elements $\phi_1, \ldots, \phi_n \in Hom_R(M, R)$ such that $\phi_i(m_i) \neq 0$ for all $i$ and $\phi_i(m_j) = 0$ for all $i \neq j$. Show that $(m_1, \ldots m_n)$ is a basis of $M$ and $(\phi_1, \ldots, \phi_n)$ is a basis of $Hom_R(M, R)$.